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Alternative spatial behaviour in the incompressible linear elastic prismatic constrained cylinder

R. J. KNOPS
Department of Mathematics, Heriot-Watt University, Edinburgh, EH14 4AS, Scotland

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Abstract. Differential inequalities are derived for two cross-sectional energy fluxes. Integration establishes exponential growth and decay estimates for the cross-sectional mean square displacement, displacement gradient and pressure. The conclusions relate to Saint–Venant's principle for an incompressible elastic cylinder and more generally to the Phragmén–Lindelöf principle and Liouville's theorem. Other contributions to the literature are briefly noted.

Key words: incompressible linear elasticity, spatial behaviour, Saint–Venant's principle.

1. Introduction

This paper considers the problem of a prismatic cylinder constrained at zero displacement along its lateral sides and composed of a linear, homogeneous, isotropic, incompressible elastic material. Data, in a sense to be precisely defined, is prescribed on the base, but no *a priori* assumptions are required on the longitudinal asymptotic behaviour of the equilibrium displacement, the (unknown) pressure or displacement gradients. The primary purpose is to establish conditions which lead to estimates for the spatial growth and decay with respect to axial distance of certain cross-sectional measures and also of the energy in a volume of the cylinder. The decay estimates relate to the familiar Saint–Venant's principle, while the more complete alternative behaviour locates the treatment within the general context of a Phragmén–Lindelöf principle.

The estimates, derived in terms of two cross-sectional energy fluxes, are independent of the cylinder's length which may be finite, semi-infinite or infinite. For simplicity, attention is almost exclusively confined to the semi-infinite cylinder, although the infinite case is briefly considered when the spatial behaviour is analogous to Liouville's theorem. Modification for the finite cylinder will become self-evident. The approach, briefly described in [1], continues the development presented in *e.g.* [2–5], and is an important variant of the method employing energy volume measures as the principal quantity. Such measures, to render the analysis meaningful, usually are assumed bounded which conceals conditions pertaining to growth. Of course, while the present primary purpose is to derive conditions for the alternative behaviour of either growth or decay, the major portion of the paper inevitably is devoted to a discussion of decay estimates. Even so, the approach produces new rates of growth and decay, and furthermore provides decay estimates for the cross-sectional mean square of displacement, pressure, displacement gradient, stress and for the energy flux.

All estimates derived involve exponential rates of growth or decay that depend only upon the geometry of the cross-section. The corresponding amplitudes are a certain linear combination of the total energy and its first-order analogue, for which upper bounds may be derived in terms of the displacement and its normal derivative when specified (pointwise) over the base. An alternative derivation of the decay estimates, that still employs cross-sectional measures, avoids the introduction of a first-order energy and leads to a simplified decay rate and an amplitude proportional only to the total energy which now may be bounded above by the mean square displacement and its tangential derivatives over the base.

Previous allied studies of the elastic cylinder, surveyed in [6–9], are mainly confined to compressible materials and apart from those already noted, to the derivation of decay estimates. The decay rates, however, degenerate in the incompressible limit, which motivated a treatment [10] simultaneously applicable to compressible and incompressible linear elasticity. A separate investigation of the incompressible problem is given in [11]. Both studies are for the corresponding free cylinder and thus are not strictly comparable with the present analysis for the constrained cylinder. Nevertheless, the respective decay rates, although difficult to exactly compare, appear not dissimilar. A more recent study [12] investigates exactly the same problem as considered in this paper, but discusses only decay estimates. The method depends upon a second-order inequality for the cross-sectional measure here denoted by $J(x_3)$ (see Equation (3.18)). While the calculations are somewhat less involved, the decay rate is inferior to that obtained by the present analysis. Techniques similar to those developed here are applied to the problem of the free cylinder in a forthcoming paper [13]. The two-dimensional problem is treated in [14].

Other relevant investigations, reported notably in [1] and references there cited, concern the steady-state Navier–Stokes flow, and hence Stokes flow, along a three-dimensional pipe and, under restrictions on the flow, obtain decay rates which for Stokes flow are also similar to those derived here. Precise comparison is again not straightforward. (See also [15] and the cited references for the corresponding problem in plane Stokes flow).

Section 2 details the basic boundary-value problem and some of the standard inequalities subsequently required. Section 3 introduces the cross-sectional energy fluxes, discusses relevant properties and derives the differential inequalities. Their integration is undertaken in Section 4 which also establishes the conditions leading to the main growth and decay estimates. Section 5 deduces decay estimates for the cross-sectional mean-square displacement, displacement gradients, stress and pressure, and the cross-sectional energy flux. The amplitude in the decay estimate is bounded above in terms of the displacement and its normal derivative specified over the cylinder's base. When the amplitude involves only the total energy, then the displacement alone needs to be prescribed. A brief summary together with some suggested applications are included in the final section.

Throughout, the existence of a suitably smooth solution is assumed. The standard summation convention is adopted, along with a comma to denote spatial partial differention. Latin and greek subscripts have the respective ranges 1, 2, 3, and 1, 2.

2. Specification of problem. Basic inequalities

Let Ω denote a semi-infinite prismatic cylinder of uniform bounded cross-section D whose boundary is sufficiently smooth to admit applications of the divergence theorem. Rectangular cartesian axes are selected with origin in the base $D(0)$ of the cylinder and with the x_3-axis parallel to the generators, so that Ω is given by

$$\Omega = \{x : (x_1, x_2) \in D, 0 \leqslant x_3 < \infty\}. \tag{2.1}$$

Let $D(x_3)$ indicate the cross-section at a distance x_3 from the base, and let $\Omega(z, z+h)$, $z \geqslant 0$, $h > 0$ denote the portion of Ω contained between $D(z)$ and $D(z+h)$. Hence,

$$\Omega(z, z+h) = \{x : (x_1, x_2) \in D, z \leqslant x_3 \leqslant x_3 + h\}. \tag{2.2}$$

It is assumed that Ω is occupied by an incompressible, homogeneous linear elastic material maintained in equilibrium under zero body-force and with the lateral sides of the cylinder held fixed at zero displacement. The base of the cylinder is loaded in a sense defined later, while the behaviour for asymptotically large values of the axial variable is not prescribed *a priori* but is to be determined.

Accordingly, the displacement $u(x)$ satisfies the equilibrium equation

$$\mu u_{i,jj} = p_{,i}\,, \quad x \in \Omega, \tag{2.3}$$

where $p(x)$ is the unknown hydrostatic pressure and μ the shear modulus; the incompressibility condition becomes

$$u_{i,i} = 0, \quad x \in \Omega; \tag{2.4}$$

and the lateral boundary conditions are

$$u_i = 0, \quad x \in \partial D \times (0, \infty). \tag{2.5}$$

It is assumed that on the base of the cylinder the distribution of displacement and tractions does not produce singularities at the boundary ∂D. Additionally it is assumed for simplicity that over the base there holds

$$\int_{D(0)} u_3(x_1, x_2, 0)\, \mathrm{d}x_1\, \mathrm{d}x_2 = 0. \tag{2.6}$$

From (2.4), (2.5) and the divergence theorem it follows that:

$$\int_{D(x_3)} u_3\, \mathrm{d}x_1\, \mathrm{d}x_2 = \int_{D(0)} u_3\, \mathrm{d}x_1\, \mathrm{d}x_2 = 0. \tag{2.7}$$

The following inequalities, stated without proof, are required in the subsequent calculations. In both inequalities, $w(x)$ is a continuously differentiable function on a plane (Lipschitz) domain D.

2.1. POINCARÉ'S INEQUALITY

Let

$$w(x) = 0, \quad x \in \partial D. \tag{2.8}$$

Then

$$\lambda_1 \int w^2\, \mathrm{d}x_1\, \mathrm{d}x_2 \leqslant \int_D w_{,\alpha} w_{,\alpha}\, \mathrm{d}x_1\, \mathrm{d}x_2, \tag{2.9}$$

where λ_1 is the first eigenvalue for the fixed membrane problem for D; *i.e.*

$$\Phi_{,\alpha\alpha} + \lambda\Phi = 0, \quad x \in D, \tag{2.10}$$

$$\Phi = 0, \quad x \in \partial D. \tag{2.11}$$

A lower bound for λ_1 is provided by the Faber–Krahn estimate [16, 17]

$$\lambda_1 \geqslant \pi j_0^2/|D|, \tag{2.12}$$

where $|D|$ is the area of D and j_0 is the smallest positive zero of the Bessel function $J_0(x)$.

When, in addition to condition (2.8), the function $w(x)$ also satisfies

$$\int_D w \,\mathrm{d}x_1 \,\mathrm{d}x_2 = 0, \tag{2.13}$$

then

$$\lambda_2 \int_D w^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2 \leqslant \int_D w_{,\alpha} w_{,\alpha} \,\mathrm{d}x_1 \,\mathrm{d}x_2, \tag{2.14}$$

where λ_2 is the smallest positive eigenvalue of the problem

$$\Phi_{,\alpha\alpha} + \lambda\Phi = k, \quad x \in D, \tag{2.15}$$

$$\Phi = 0, \quad x \in \partial D, \tag{2.16}$$

$$\int_D \Phi \,\mathrm{d}x_1 \,\mathrm{d}x_2 = 0, \tag{2.17}$$

for constant k. It follows that $\lambda_1 \leqslant \lambda_2$; a further estimate for λ_2 is discussed in *e.g.*, [18].

2.2. BABUSKA–AZIZ THEOREM [19]

Suppose $w(x)$ satisfies

$$\int_D w \,\mathrm{d}x_1 \,\mathrm{d}x_2 = 0. \tag{2.18}$$

Then there exists a differentiable vector function ψ_α such that

$$\psi_{\alpha,\alpha} = w, \quad x \in D, \tag{2.19}$$

$$\psi_\alpha = 0, \quad x \in \partial D,$$

and a positive constant C depending only on the geometry of D such that

$$\int_D \psi_{\alpha,\beta}\psi_{\alpha,\beta} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \leqslant C \int_D (\psi_{\alpha,\alpha})^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2. \tag{2.20}$$

Various bounds for the constant C are derived in [10], where it is also shown that $C \geqslant 1$, with the optimal value $C = 1$ attained for a circle of arbitrary radius.

3. Energy-flux functions and the associated differential inequalities

The behaviour of the displacement on Ω is obtained in suitable measure from that of two functions shown in this section to satisfy two differential inequalities. These two inequalities are optimally combined into a single one whose integration, undertaken in the next section, leads to estimates on the behaviour of the energy contained in $\Omega(z, \infty)$ and hence on the cross-sectional mean square measure of the displacement. Both functions used in the inequalities are related to energy flux across a cross-section of the cylinder.

PROPOSITION 3.1. *The energy-flux function, defined by*

$$H(x_3) = \int_{D(x_3)} (\mu u_i u_{i,3} - p u_3)\, \mathrm{d}x_1\, \mathrm{d}x_2, \tag{3.1}$$

satisfies for all $x_3 \geqslant 0$ *the condition*

$$H'(x_3) = \mu \int_{D(x_3)} u_{i,j} u_{i,j}\, \mathrm{d}x_1\, \mathrm{d}x_2 \geqslant 0, \tag{3.2}$$

where a superposed prime denotes differentiation with respect to the axial variable x_3. *Furthermore,* $H(x_3)$ *satisfies the inequality*

$$|H(x_3)| \leqslant \frac{\mu}{2} \int_{D(x_3)} \{a u_{\alpha,\beta} u_{\alpha,\beta} + b u_{3,\alpha} u_{3,\alpha} + d u_{\alpha,3} u_{\alpha,3} + e u_{3,3}^2 + f u_{\alpha,33} u_{\alpha,33}\}\, \mathrm{d}x_1\, \mathrm{d}x_2, \quad x_3 \geqslant 0, \tag{3.3}$$

where

$$a = c_1 \lambda_1^{-1/2} + c_4^{-1} C^{1/2} \lambda_2^{-1/2}, \tag{3.4}$$

$$b = c_2 \lambda_2^{-1/2} + c_3 C^{1/2} (\lambda_1 \lambda_2)^{-1/2} + c_4 C^{1/2} \lambda_2^{-1/2}, \tag{3.5}$$

$$d = c_1^{-1} \lambda_1^{-1/2}, \tag{3.6}$$

$$e = c_2^{-1} \lambda_2^{-1/2}, \tag{3.7}$$

$$f = c_3^{-1} C^{1/2} (\lambda_1 \lambda_2)^{-1/2}, \tag{3.8}$$

and $c_i (> 0) i = 1, \dots 4$ *are positive constants to be determined later. The constants* λ_1, λ_2, C *are those appearing in* (2.9), (2.14) *and* (2.20), *respectively.*

Proof. Let $h > 0$ be an arbitrary positive constant and let $E(x_3, x_3 + h)$ denote the energy contained in that portion of the cylinder enclosed between the cross-sections at x_3 and $x_3 + h$ from the base. Thus

$$E(x_3, x_3 + h) = \mu \int_{\Omega(x_3, x_3+h)} u_{i,j} u_{i,j}\, \mathrm{d}x. \tag{3.9}$$

The equilibrium equations, together with (2.4), (2.5), and the divergence theorem then yield the identity

$$H(x_3 + h) - H(x_3) = E(x_3, x_3 + h), \tag{3.10}$$

from which (3.2) follows on differentiation.

To establish inequality (3.3), it is convenient to first separately consider each term on the right side of (3.1). Application of Schwarz's inequality, followed by the Poincaré inequalities (2.9) and (2.14) gives

$$
\begin{aligned}
&\left|\int_{D(x_3)} u_i u_{i,3}\,\mathrm{d}x_1\,\mathrm{d}x_2\right| \\
&\quad = \left|\int_{D(x_3)} \left(u_\alpha u_{\alpha,3} + u_3 u_{3,3}\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\right| \\
&\quad \leqslant \left(\int_{D(x_3)} u_\alpha u_\alpha\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(x_3)} u_{\alpha,3}u_{\alpha,3}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \\
&\qquad + \left(\int_{D(x_3)} u_3^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(x_3)} u_{3,3}^2\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \\
&\quad \leqslant \left(\lambda_1^{-1}\int_{D(x_3)} u_{\alpha,\beta}u_{\alpha,\beta}\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(x_3)} u_{\alpha,3}u_{\alpha,3}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2} \\
&\qquad + \left(\lambda_2^{-1}\int_{D(x_3)} u_{3,\alpha}u_{3,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(x_3)} u_{3,3}^2\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}. \qquad (3.11)
\end{aligned}
$$

To treat the second term, note that (2.7) and the Babuska–Aziz inequality imply for each fixed x_3 the existence of a differentiable vector function $\psi_\alpha(x_1, x_2)$ such that $\psi_{\alpha,\alpha} = u_3$ in D, and $\psi_\alpha = 0$ on ∂D. Hence,

$$
-\int_{D(x_3)} p\; u_3\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{D(x_3)} p\psi_{\alpha,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2 \qquad (3.12)
$$

$$
= \mu\int_{D(x_3)} \psi_\alpha u_{\alpha,33}\,\mathrm{d}x_1\,\mathrm{d}x_2 - \mu\int_{D(x_3)} \psi_{\alpha,\beta}u_{\alpha,\beta}\,\mathrm{d}x_1\,\mathrm{d}x_2, \qquad (3.13)
$$

where the divergence theorem and (2.3) have been used. By virtue of Schwarz's inequality, the Poincaré inequality (2.9) and the Babuska–Aziz inequality (2.20), it next follows that

$$
\begin{aligned}
&-\int_{D(x_3)} p\; u_3\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad \leqslant \mu\left[\left(\int_{D(x_3)} \psi_\alpha\psi_\alpha\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(x_3)} u_{\alpha,33}u_{\alpha,33}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}\right. \\
&\qquad\quad \left. + \left(\int_{D(x_3)} \psi_{\alpha,\beta}u_{\alpha,\beta}\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(x_3)} u_{\alpha,\beta}u_{\alpha,\beta}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}\right] \\
&\quad \leqslant \mu\left(\int_{D(x_3)} \psi_{\alpha,\beta}\psi_{\alpha,\beta}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}
\end{aligned}
$$

$$\times \left[\left(\lambda_1^{-1} u_{\alpha,33} u_{\alpha,33} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/2} + \left(\int_{D(x_3)} u_{\alpha,\beta} u_{\alpha,\beta} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/2} \right]$$

$$\leqslant \mu \left(C \int_{D(x_3)} u_3^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{\frac{1}{2}}$$

$$\times \left[\left(\lambda_1^{-1} \int_{D(x_3)} u_{\alpha,33} u_{\alpha,33} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/2} + \left(\int_{D(x_3)} u_{\alpha,\beta} u_{\alpha,\beta} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/2} \right]$$

$$\leqslant \mu \left(C \lambda_2^{-1} \int_{D(x_3)} u_{3,\alpha} u_{3,\alpha} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/2}$$

$$\times \left[\left(\lambda_1^{-1} \int_{D(x_3)} u_{\alpha,33} u_{\alpha,33} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/2} + \left(\int_{D(x_3)} u_{\alpha,\beta} u_{\alpha,\beta} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right)^{1/2} \right]. \qquad (3.14)$$

The insertion of (3.11) and (3.14) into (3.1) together with an application of the arithmetic-geometric mean inequality then leads easily to (3.3).

PROPOSITION 3.2. *The first-order energy flux, defined by*

$$L(x_3) = \mu \int_{D(x_3)} (u_{\alpha,3} u_{\alpha,33} + u_{3,\alpha} u_{3,\alpha 3}) \,\mathrm{d}x_1 \,\mathrm{d}x_2, \qquad (3.15)$$

satisfies for all $x_3 \geqslant 0$ *the conditions*

(i) $L(x_3) = \frac{1}{2} J'(x_3)$, *where* $J(x_3) = \mu \int_{D(x_3)} (u_{\alpha,3} u_{\alpha,3} + u_{3,\alpha} u_{3,\alpha}) \,\mathrm{d}x_1 \,\mathrm{d}x_2$, (3.16)

(ii) $L'(x_3) = \mu \int_{D(x_3)} u_{i,j3} u_{i,j3} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \geqslant 0$, (3.17)

(iii)
$$\nu \,|L(x_3)| \leqslant \frac{\mu}{2} \int_{D(x_3)} \big[c_5 u_{\alpha,3} u_{\alpha,3} + c_6 u_{3,\alpha} u_{3,\alpha} + \nu^2 c_5^{-1} u_{\alpha,33} u_{\alpha,33} + \nu^2 c_6^{-1} \, u_{3,\alpha 3} u_{3,\alpha 3} \big] \,\mathrm{d}x_1 \,\mathrm{d}x_2, \qquad (3.18)$$

where ν, c_5, c_6 *are arbitrary positive constants to be chosen later.*

Proof. Condition (i) is obvious.

To establish (ii), let $E_1(x_3, x_3 + h)$ denote the first order energy contained in $\Omega(x_3, x_3 + h)$, where h is a positive constant. Thus, by definition

$$E_1(x_3, x_3 + h) = \mu \int_{\Omega(x_3, x_3 + h)} u_{i,j3} u_{i,j3} \,\mathrm{d}x. \qquad (3.19)$$

As before, the equilibrium equations, (2.4), (2.5), the divergence theorem together with additional appeal to the prismatic property of the cylinder, lead to the identity:

$$L(x_3 + h) - L(x_3) = E_1(x_3, x_3 + h), \qquad (3.20)$$

which on differentiation yields expression (3.17).

Successive application of the Schwarz and arithmetic-geometric mean inequalities in (3.15) enables inequality (3.18) to be easily proved.

PROPOSITION 3.3. *Let the function $G(x_3)$ be defined by*

$$G(x_3) = H(x_3) + \nu L(x_3), \tag{3.21}$$

where ν is a positive constant to be determined. Then $G(x_3)$ satisfies the inequality

$$|G(x_3)| \leqslant \frac{\mu}{2}\int_{D(x_3)} \left\{ a u_{\alpha,\beta}u_{\alpha,\beta} + (b+c_6)u_{3,\alpha}u_{3,\alpha} + (d+c_5)u_{\alpha,3}u_{\alpha,3} + e u_{3,3}^2 \right.$$
$$\left. + (f+\nu^2 c_5^{-1})u_{\alpha,33}u_{\alpha,33} + \nu^2 c_6^{-1} u_{3,\alpha 3}u_{3,\alpha 3}\right\} \mathrm{d}x_1\,\mathrm{d}x_2, \tag{3.22}$$

where the constants a, b, d, e, f are given by (3.4)–(3.8), *respectively.*

Proof. Inequality (3.22) follows immediately from (3.3) and (3.18).

The constants c_i (i = 1, …, 6) and ν are now chosen such that

$$a = (b+c_6) = (d+c_5) = e, \tag{3.23}$$

$$(f+\nu^2 c_5^{-1}) = \nu^2 c_6^{-1} = \nu e. \tag{3.24}$$

A lengthy but straightforward calculation shows that an appropriate choice gives the values

$$e = (Q/2\lambda_2)^{1/2}, \tag{3.25}$$

$$\nu = (C/\lambda_1\lambda_2)^{1/2}, \tag{3.26}$$

where

$$Q = \left(\frac{\lambda_2}{\lambda_1} + C + 1\right) + \sqrt{\left[\left(\frac{\lambda_2}{\lambda_1} + C - 1\right)^2 + 4C\right]} + (3+\sqrt{5})\left(C\frac{\lambda_2}{\lambda_1}\right)^{1/2}. \tag{3.27}$$

The following theorem is therefore proved on recalling (3.2) and (3.17).

THEOREM 3.1. *The function*

$$G(x_3) = H(x_3) + \nu L(x_3), \tag{3.28}$$

where $H(x_3), L(x_3), \nu$ are given by (3.1), (3.15) *and* (3.26), *satisfies the differential inequality*

$$|G(x_3)| \leqslant \frac{e}{2}G', \quad x_3 \geqslant 0, \tag{3.29}$$

where the constant e is given by (3.25).

The integration of (3.29) is discussed in the next section.

Finally, this Section considers properties of the function $J(x_3)$ defined in (3.16). Note that $J(x_3)$ has also been introduced in [20], [6] to study the corresponding decay behaviour of a laterally constrained prismatic cylinder composed of a linear isotropic compressible elastic material and maintained in equilibrium by data specified over the base with the displacement gradients asymptotically vanishing for large values of the axial variable. See also [12] for a treatment of the incompressible problem. It is shown that $J(x_3)$ is not only convex, as here follows from (3.17), but also that $J^{1/2}(x_3)$ satisfies the condition of 'generalised' convexity. For present purposes, it is sufficient to prove that $J^{1/2}(x_3)$ is convex. Thus, Schwarz's inequality applied to (3.15) gives

$$\begin{aligned}\frac{1}{2}\left(J'\right)^2 &\leqslant 2\mu^2\int_{D(x_3)}\left(u_{\alpha,3}u_{\alpha,3}+u_{3,\alpha}u_{3,\alpha}\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &\quad\times\int_{D(x_3)}\left(u_{\alpha,33}u_{\alpha,33}+u_{3,\alpha 3}u_{3,\alpha 3}\right)\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &\leqslant JJ'',\end{aligned} \tag{3.30}$$

where (3.16) and (3.17) have been used. It follows immediately that

$$\left[J^{1/2}(x_3)\right]'' \geqslant 0, \quad x_3 \geqslant 0, \tag{3.31}$$

provided $J(x_3) \neq 0$, $x_3 \geqslant 0$.

4. Integration of main inequality

Integration of the main differential inequality (3.29) establishes the existence of alternative behaviour on Ω similar to the classical Phragmén–Lindelöf theorem in potential theory. Furthermore, since (3.29) depends only on the variable x_3, it is valid irrespective of the length of the cylinder and hence in particular holds on a (prismatic) cylinder of infinite length. In this case, it is shown that only the trivial displacement exists in the class of bounded energies, a result analogous to Liouville's theorem.

Conclusions are derived in a succession of Propositions.

PROPOSITION 4.1. *Suppose conditions on the cylinder's base are such that*

$$G(0) > 0. \tag{4.1}$$

Then $G(x_3)$ *satisfies the following estimate*

$$G(x_3) \geqslant \mathrm{e}^{\gamma x_3}G(0), \quad x_3 \geqslant 0. \tag{4.2}$$

where $\gamma = 2/e$, *and* e *is given by* (3.25).

Proof. By properties (3.2), (3.17), it follows from (4.1) that

$$G(x_3) \geqslant 0, \quad x_3 \geqslant 0. \tag{4.3}$$

Hence, the appropriate component of inequality (3.29) gives

$$\gamma G(x_3) \leqslant G'(x_3), \quad x_3 \geqslant 0, \tag{4.4}$$

which on integration yields (4.2).

REMARK 4.1. Condition (4.1) together with (3.10) and (3.20) implies the unboundedness of energies on Ω. That is

$$\lim_{h\to\infty} \{E(x_3, x_3+h) + \nu E_1(x_3, x_3+h)\}$$

does not exist.

Condition (4.1) is not the only one inducing energies to become unbounded on $\Omega(x_3, z)$ as $z \to \infty$. Indeed, let $z_1 > 0$, $z_2 > 0$ and suppose

$$H(z_1) > 0, \quad L(z_2) > 0. \tag{4.5}$$

Then $G(z) > 0$, where $z = \max\ (z_1, z_2)$, and hence (4.4) holds for $x_3 \geqslant z$. Integration immediately gives

$$G(x_3) \geqslant G(z) \exp \gamma(x_3 - z), \quad x_3 \geqslant z, \tag{4.6}$$

and the conclusion follows.

Less trivial conditions yielding unbounded energies are provided by the next two results.

PROPOSITION 4.2. *Let $z > 0$ be fixed, and suppose that $H(z) > 0$. Furthermore, suppose that $L(x_3) \leqslant 0$, for all $x_3 \geqslant 0$.*

Then the limit

$$\lim_{h\to\infty} E(x_3, x_3+h) \tag{4.7}$$

does not exist.

Proof. By (3.2), $H(x_3) > 0$ for $x_3 \geqslant z$, and hence from (3.3) it may be concluded that

$$\begin{aligned} H(x_3) \ \leqslant\ & \frac{\mu}{2}\int_{D(x_3)} \left(au_{\alpha,\beta}u_{\alpha,\beta} + bu_{3,\alpha}u_{3,\alpha} + du_{\alpha,3}u_{\alpha,3} + eu_{3,3}^2\right) \mathrm{d}x_1\,\mathrm{d}x_2 \\ & + \frac{f}{2}L'(x_3), \end{aligned} \tag{4.8}$$

where the constants a, b, d, e, f are given by (3.4)–(3.8) but with c_i, $i = 1, \ldots, 4$ as yet unspecified.

Set

$$c_4 = \tfrac{1}{2}c_2^{-1}C^{-1/2}, \qquad c_1 = c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{1/2}, \qquad c_2 = \left(\frac{\lambda_2}{\lambda_1} + 2C\right)^{-1/2}, \tag{4.9}$$

and let c_3 be chosen sufficiently large to satisfy

$$c_3 \geqslant \left(\frac{\lambda_1}{C}\right)^{1/2}\left(\frac{\lambda_2}{2\lambda_1} + C - 1\right)c_2^{-1}. \tag{4.10}$$

Then $a = d = e \leqslant b$ and (4.8) becomes

$$H(x_3) \leqslant \frac{b}{2} H'(x_3) + \frac{f}{2} L'(x_3), \quad x_3 \geqslant z, \tag{4.11}$$

which, since $L(x_3) \leqslant 0$, $x_3 \geqslant 0$, may be re-written

$$0 \leqslant \left(e^{-\delta x_3} H\right)' + \frac{f}{b} \left(e^{-\delta x_3} L\right)', \quad x_3 \geqslant z, \tag{4.12}$$

where $\delta = 2/b$. Integration of (4.12) then gives

$$e^{\delta(x_3 - z)} \left[H(z) + \frac{f}{b} L(z)\right] \leqslant H(x_3), \quad x_3 \geqslant z, \tag{4.13}$$

and the conclusion follows from (3.10) provided that

$$bH(z) + fL(z) > 0. \tag{4.14}$$

Condition (4.14) can always be ensured by choosing c_3 sufficiently large.

PROPOSITION 4.3. *Let $z > 0$ be fixed and suppose that $L(z) > 0$. Suppose further that $H(x_3) \leqslant 0$, $x_3 \geqslant 0$. Then the limit*

$$\lim_{h \to \infty} E_1(x_3, x_3 + h) \tag{4.15}$$

does not exist.

Proof. It follows by hypothesis and (3.17) that $L(x_3) > 0$, $x_3 \geqslant z$. Hence, inequality (3.18) leads to

$$L(x_3) \leqslant \frac{c_5}{2} H'(x_3) + \frac{c_5^{-1}}{2} L'(x_3), \quad x_3 \geqslant z, \tag{4.16}$$

where the choice $\nu = 1$, $c_5 = c_6$ has been taken, so that

$$0 \leqslant c_5^2 \left(e^{-c_5 x_3} H\right)' + \left(e^{-c_5 x_3} L\right)', \quad x_3 \geqslant z. \tag{4.17}$$

Integration gives

$$e^{c_5(x_3 - z)} \left[c_5^2 H(z) + L(z)\right] \leqslant L(x_3), \quad x_3 \geqslant z, \tag{4.18}$$

which implies the conclusion on taking c_5 to be sufficiently small.

REMARK 4.2. The conditions of Proposition 4.3 also imply the nonexistence of the limit (4.7). This result is proved by noting that the function $J(x_3)$, defined in (3.16), satisfies

$$0 \leqslant J(x_3) \leqslant H'(x_3), \quad x_3 \geqslant 0, \tag{4.19}$$

By hypothesis $L(x_3) > 0$, $x_3 \geqslant z$, and hence $J(x_3) > J(z)$. Insertion into (4.19) then gives after an integration

$$(x_3 - z) J(z) < E(x_3, z), \quad x_3 \geqslant z, \tag{4.20}$$

and the conclusion follows.

Indeed, a slightly improved result is possible:

PROPOSITION 4.4. *Let $z > 0$ be fixed and suppose $L(z) > 0$. Then the limit* (4.7) *does not exist.*

Proof. Since $L(z) > 0$ implies $L(x_3) > 0, x_3 \geqslant z$, there exists $z_1 \geqslant z$ such that $J(x_3) > 0, x_3 \geqslant z_1$. Hence (3.31) is valid on $x_3 \geqslant z_1$, and integration yields

$$J(x_3) \geqslant \left[J(z_1) + \frac{(x_3 - z_1)}{2} J'(z_1) \right]^2 / J(z_1). \tag{4.21}$$

But $J(x_3) \leqslant H'(x_3)$ and a further integration proves the Proposition.

The results so far describe conditions under which the solution in various measures possesses a growth behaviour and for which the energies $E(x_3, \infty)$, $E_1(x_3, \infty)$ are unbounded for any $x_3 \geqslant 0$. An immediate implication is that solutions in the class of bounded energies E, E_1 must satisfy the conditions

$$H(x_3) < 0, \quad L(x_3) < 0, \quad x_3 \geqslant 0. \tag{4.22}$$

Inequality in (4.22) is strict since if $H(z_1) = 0$ for some $z_1 > 0$, then $H(x_3) \equiv 0$ for $x_3 \geqslant z_1$ otherwise $H(z_3) > 0$ for $x_3 > z_1$ and Proposition 4.2 contradicts the assumed boundedness of E. Hence by (3.2) and the boundary conditions, the displacement is identically zero. Similarly, if $L(z_2) = 0$ for some $z_2 > 0$, then $L(z_3) = 0$, for $x_3 \geqslant z_2$, otherwise $L(x_3) > 0$ for $x_3 > z_2$ and Proposition 4.4 provides a contradiction. It then follows from (3.17) and the boundary conditions that $u_i = u_i(x_\alpha)$ which must again be identically zero in the class of solutions with bounded energy E. Strict inequality in (4.22) excludes these trivial cases.

The next series of results establishes various decay estimates for solutions in the class of bounded energies. It is convenient to introduce the notation

$$E(x_3) \equiv E(x_3, \infty), \qquad E_1(x_3) \equiv E_1(x_3, \infty). \tag{4.23}$$

THEOREM 4.1. *In the class of bounded energies $E(0)$, $E_1(0)$, the solution to* (2.3)–(2.6) *satisfies the decay estimate*

$$-G(x_3) \leqslant -G(0)\mathrm{e}^{-\gamma x_3}, \quad x_3 \geqslant 0, \tag{4.24}$$

where the function $G(x_3)(< 0)$ is defined by (3.28), *the positive constant $\gamma = 2/e$, and e is given by* (3.25).

Proof. Since the energies are assumed bounded, it follows that (4.22) holds, or $H(x_3) = L(x_3) = 0$ for some $x_3 \geqslant z > 0$, when (4.24) is trivially satisfied. Hence, $G(x_3) < 0, x_3 \geqslant 0$, and thus (3.29) yields

$$-\gamma G(x_3) \leqslant G'(x_3), \quad x_3 \geqslant 0, \tag{4.25}$$

from which (4.24) follows by integration.

In the form (4.24), the decay estimate does not readily provide useful information and hence the next section of this paper is devoted to the derivation from (4.24) of more meaningful decay estimates.

REMARK 4.3. Since the derivation of the basic inequality (3.29) involves only cross-sectional integrals, its validity is unaffected by the length of cylinder and therefore (3.29) applies to a cylinder of finite and infinite length. Integration of (3.29) in these circumstances is accomplished in the manner already described. As an example, consider the cylinder of infinite length. It may easily be shown that the only displacement existing in the class of bounded energies $E(0)$, $E_1(0)$, identically vanishes. Thus, suppose for arbitrary z, $G(z) \neq 0$. Let $G(z) > 0$. Then (4.6) establishes a contradiction. Thus, let $G(z) < 0$. Then $G(x_3) \leqslant G(z) < 0$, $x_3 \leqslant z$, and hence (4.25) holds for $x_3 \leqslant z$. Integration gives

$$-G(x_3) \geqslant -G(z)\mathrm{e}^{\gamma(z-x_3)}, \quad x_3 \leqslant z, \tag{4.26}$$

which again leads to a contradiction. Thus, $G(x_3) \equiv 0, -\infty < x_3 < \infty$, from which the conclusion follows.

5. Deductions from the basic decay estimate

Inequality (4.24) implies that

$$\lim_{x_3\to\infty} H(x_3) = \lim_{x_3\to\infty} L(x_3) = 0, \tag{5.1}$$

and on recalling (3.10), (3.20) and the definitions (4.23), it follows that (4.24) may be equivalently written as

$$E(x_3) + \nu E_1(x_3) \leqslant [E(0) + \nu E_1(0)]\mathrm{e}^{-\gamma x_3}, \quad x_3 \geqslant 0, \tag{5.2}$$

which in turn implies that

$$\lim_{x_3\to\infty} \int_{D(x_3)} u_{i,j}u_{i,j}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \lim_{x_3\to\infty} \int_{D(x_3)} u_{i,j3}u_{i,j3}\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0. \tag{5.3}$$

Poincaré's inequalities then show that

$$\lim_{x_3\to\infty} \int_{D(x_3)} u_i u_i\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0,$$

and hence

$$\begin{aligned}
\int_{D(x_3)} u_i u_i\,\mathrm{d}x_1\,\mathrm{d}x_2 &= -2\int_{\Omega(x_3,\infty)} u_i u_{i,3}\,\mathrm{d}x \\
&\leqslant 2\left(\int_{\Omega(x_3,\infty)} u_i u_i\,\mathrm{d}x \int_{\Omega(x_3,\infty)} u_{i,3}u_{i,3}\,\mathrm{d}x\right)^{1/2} \\
&\leqslant \left(\lambda_1^{1/2}\mu\right)^{-1} E(x_3) \\
&\leqslant \left(\lambda_1^{1/2}\mu\right)^{-1} [E(0) + \nu E_1(0)]\,\mathrm{e}^{-\gamma x_3},
\end{aligned} \tag{5.4}$$

by virtue of Schwarz's inequality, the arithmetic-geometric mean inequality and (5.2). Hence, the displacement in cross-sectional mean square measure has an exponentially decreasing upper bound which provides the alternative behaviour in a Phragmén–Lindelöf principle. A similar calculation shows that

$$\int_{D(x_3)} u_{i,j}u_{i,j}\,\mathrm{d}x_1\,\mathrm{d}x_2 \leqslant 2\,[E(0)+\nu E_1(0)]\,\mathrm{e}^{-\gamma x_3}, \quad x_3 \geqslant 0. \tag{5.5}$$

Let the linear strain and rotation be defined as usual by

$$e_{ij} = \tfrac{1}{2}\left(u_{i,j}+u_{j,i}\right), \quad \omega_{ij} = \tfrac{1}{2}\left(u_{i,j}-u_{j,i}\right). \tag{5.6}$$

Then

$$u_{i,j}u_{i,j} = e_{ij}e_{ij} + \omega_{ij}\omega_{ij}, \tag{5.7}$$

so that (5.5) yields corresponding decay estimates for the cross-sectional mean squares of the strain and rotation.

An estimate for the pressure is obtained from an inequality derived in [21] which when modified to the present circumstances becomes

$$\int_{D(x_3)} p^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \leqslant k_1\int_{D(x_3)} u_iu_i\,\mathrm{d}x_1\,\mathrm{d}x_2 + k_2\int_{D(x_3)} u_{i,\alpha}u_{i,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{5.8}$$

where k_1, k_2, are computable positive constants. Substitution of (5.4) and (5.5) then leads to

$$\int_{D(x_3)} p^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \leqslant \left[k_1\left(\lambda_1^{1/2}\mu\right)^{-1} + 2k_2\right][E(0)+\nu E_1(0)]\,\mathrm{e}^{-\gamma x_3}, \quad x_3 \geqslant 0. \tag{5.9}$$

A decay estimate for the cross-sectional mean square of the stress, defined to be

$$\sigma_{ij} = 2\mu e_{ij} - p\delta_{ij}, \tag{5.10}$$

where δ_{ij} is the Kronecker delta, follows immediately from (5.5), (5.7) and (5.9).

A pointwise estimate for the displacement may be derived by noting from (2.3) that $u_i(x)$ is biharmonic and then employing a mean-value theorem which states:

LEMMA [22]. *Let $v(x)$ be a biharmonic real valued function in the three-dimensional region Λ. Suppose $v \in L^2(\Lambda)$, and let d be the shortest distance of the point $x \in \Lambda$ from $\partial\Lambda$. Then the following estimate holds*:

$$|v(x)| \leqslant 1{\cdot}915 d^{-3/2}\left(\int_\Lambda |v(y)|^2\,\mathrm{d}y\right)^{1/2} \tag{5.11}$$

The proof uses spherical means to express the pointwise value of the biharmonic function in terms of its average values over the volume of two spheres, one dilated from the other. Minimisation of the dilation factor leads to the multiplicative constant in (5.11).

To apply the lemma, consider the point $x \in \Omega$, and select z to satisfy $0 \leqslant z \leqslant x_3$. Then $x \in \Omega(z,\infty)$, and (5.11) gives

$$|u_i(x)|^2 \leqslant (1{\cdot}915)^2 d^{-3}\int_{\Omega(z,\infty)} u_j(y)u_j(y)\,\mathrm{d}y, \tag{5.12}$$

where d now denotes the shortest distance of x from $\partial\Omega(z,\infty)$. Poincaré's inequality together with (5.2) then leads to

$$\begin{aligned}|u_i(x)|^2 &\leqslant (1{\cdot}915)^2 d^{-3}\lambda_1^{-1}\int_{\Omega(z,\infty)} u_{j,\alpha}u_{j,\alpha}\,\mathrm{d}y\\ &\leqslant (1{\cdot}915)^2(d^3\lambda_1\mu)^{-1}E(z)\\ &\leqslant (1{\cdot}915)^2(d^3\lambda_1\mu)^{-1}[E(0)+\nu E_1(0)]\mathrm{e}^{-\gamma(x_3-d)},\end{aligned}$$

the required estimate for $d \leqslant \operatorname{dist}(x,\partial D)$. Otherwise consider $\Omega(x_3-d,\infty)$.

The estimate degenerates as the point x approaches a point of $\partial\Omega$, which is not unreasonable since the lemma is valid irrespective of the smoothness of the boundary. Irregular boundary points create in their neighbourhood singularities in the values of the displacement and its gradient consistent with the estimate. On the other hand, it is possible to establish a corresponding estimate valid near regular boundary points, and a proof is constructed in [22].

It remains to express the energies $E(0)$, $E_1(0)$ in terms of the data defined on the base. Only the case in which $u_i(x_{\alpha,}0)$ and $u_{i,3}(x_{\alpha,}0)$ are specified subject to (2.6) is treated.

A bound for $E(0)$ may be obtained by the method described in [2] which gives

$$E(0) \leqslant \left(\int_{D(0)} u_{i,\alpha}u_{i,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(0)} u_i u_i\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}. \tag{5.13}$$

Next, observe that since the cylinder Ω is assumed to be prismatic, (2.3)–(2.5) remain valid for $u_{i,3}(x)$ and hence repetition of the argument leading to (5.13) yields the following upper bound for $E_1(0)$:

$$E_1(0) \leqslant \left(\int_{D(0)} u_{i,3\alpha}u_{i,3\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(0)} u_{i,3}u_{i,3}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)^{1/2}. \tag{5.14}$$

The second integral on the right in (5.14) may be bounded in terms of tangential derivatives over $D(0)$ of $u_i(x_\alpha,0)$ by means of the conservation law

$$\int_{D(x_3)} \left(\sigma_{ij}u_{i,j} - 2\sigma_{i3}u_{i,3}\right)\mathrm{d}x_1\,\mathrm{d}x_2 = \text{constant}, \qquad x_3 \geqslant 0, \tag{5.15}$$

and the derived asymptotic behaviour (5.3), (5.9), where σ_{ij} is given by (5.10). It then follows that

$$\begin{aligned}\mu\int_{D(x_3)} u_{\alpha,3}u_{\alpha,3}\,\mathrm{d}x_1\,\mathrm{d}x_2 &= -\mu\int_{D(x_3)} u_{\alpha,\alpha}u_{\beta,\beta}\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &\quad +\mu\int_{D(x_3)} u_{i,\alpha}u_{i,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2 - 2\int_{D(x_3)} pu_{\alpha,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned} \tag{5.16}$$

Hence, from (5.8) it is possible to conclude that

$$\begin{aligned}\mu\int_{D(x_3)} u_{i,3}u_{i,3}\,\mathrm{d}x_1\,\mathrm{d}x_2 &\leqslant \mu\int_{D(x_3)} u_{i,\alpha}u_{i,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &\quad +2\left[k_1\int_{D(x_3)} u_i u_i\,\mathrm{d}x_1\,\mathrm{d}x_2 + k_2\int_{D(x_3)} u_{i,\alpha}u_{i,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2\right]^{1/2}\\ &\quad \times\left[\int_{D(x_3)} u_{\alpha,\alpha}u_{\beta,\beta}\,\mathrm{d}x_1\,\mathrm{d}x_2\right]^{1/2}\end{aligned} \tag{5.17}$$

which is the desired upper bound.

Under the assumptions that $E(0)$ and $E_1(0)$ are bounded, an upper bound for $E_1(z)$ in terms of $E(0)$ may be obtained, where $z > 0$. These assumptions, as already shown, imply that $H(x_3) \leqslant 0$, $L(x_3) \leqslant 0$, $x_3 \geqslant 0$, and hence the previous decay estimates and asymptotic behaviour, are valid. Therefore it follows from (3.16), (3.20) that

$$2E_1(x_3) = -J'(x_3), \quad x_3 \geqslant 0, \tag{5.18}$$

and so

$$2(x_3 - z)E_1(x_3) \leqslant 2\int_z^{x_3} E_1(\eta)\,\mathrm{d}\eta = -J(x_3) + J(z) \leqslant E(z), \quad x_3 \geqslant z \geqslant 0.$$

Hence, it may be concluded that

$$E_1(x_3) \leqslant E(0)/2x_3, \quad x_3 \geqslant 0. \tag{5.19}$$

Upon integration of (4.25) over the interval (z, x_3), it is easy to prove that the previous decay estimates continue to hold with obvious modification. For example, (5.2) becomes

$$\begin{aligned} E(x_3) + \nu E_1(x_3) &\leqslant [E(z) + \nu E_1(z)]\,\mathrm{e}^{-\gamma(x_3 - z)}, \quad x_3 \geqslant z && (5.20)\\ &\leqslant E(0)\left[1 + \frac{\nu}{2z}\right]\mathrm{e}^{-\gamma(x_3 - z)}, \quad x_3 \geqslant z. && (5.21)\end{aligned}$$

Thus, provided attention is confined to an interval $[z, \infty)$, $z > 0$, the amplitude appearing in the various decay estimates depends on $E(0)$ which has been bounded in terms of the data on $D(0)$ (see (5.13)). Of course, in the limit $z \to 0$, this amplitude becomes infinite.

Explicit dependence of certain decay estimates on $E_1(0)$ may be removed, again under the assumption that $E(0)$, $E_1(0)$ are bounded. Thus, application of Schwarz's inequality together with (5.8) to (3.1) leads to

$$\begin{aligned} |H(x_3)| &\leqslant \mu\left[\int_{D(x_3)} u_i u_i \,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(x_3)} u_{i,3}u_{i,3}\,\mathrm{d}x_1\,\mathrm{d}x_2\right]^{1/2}\\ &\quad + \left[\int_{D(x_3)} p^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(x_3)} u_3^2\,\mathrm{d}x_1\,\mathrm{d}x_2\right]^{1/2}\\ &\leqslant \mu\lambda_1^{-1/2}\left[\int_{D(x_3)} u_{i,\alpha}u_{i,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2 \int_{D(x_3)} u_{i,3}u_{i,3}\,\mathrm{d}x_1\,\mathrm{d}x_2\right]^{1/2}\\ &\quad + \mu\lambda_1^{-1/2}k_3 \int_{D(x_3)} u_{i,\alpha}u_{i,\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2, \end{aligned} \tag{5.22}$$

where $\mu k_3 = \left(k_1\lambda_1^{-1/2} + k_2\right)^{1/2}$ and Poincaré's inequality has been used. The arithmetic-geometric mean inequality, with optimum choice of arbitrary constant, then gives

$$|H(x_3)| \leqslant \gamma_1^{-1}H'(x_3), \tag{5.23}$$

where

$$\gamma_1 = 2\lambda_1^{1/2}\left(\sqrt{k_3^2+1}-1\right). \tag{5.24}$$

Under the assumption that the energies $E(0)$, $E_1(0)$ are bounded, it is known that $H(x_3) \leqslant 0$, $x_3 \geqslant 0$. (Note that without this assumption the asymptotic behaviour (5.3) may not be valid and hence the inequality in [21] on which (5.8) is based may become vacuous.) Integration of (5.23) then yields

$$\begin{aligned} -H(x_3) &\leqslant -H(0)e^{-\gamma_1 x_3}, \quad x_3 \geqslant 0,\\ &= E(0)e^{-\gamma_1 x_3}, \quad x_3 \geqslant 0. \end{aligned} \tag{5.25}$$

Results similar to (5.2), $(5.3)_1$, (5.4) may be derived as before, but the amplitudes are now expressed in terms only of the energy $E(0)$.

6. Concluding remarks

New growth and decay estimates have been derived for various cross-sectional and volume measures of the equilibrium solution to the laterally constrained cylinder composed of a linear homogeneous incompressible elastic material. Decay always occurs in the class of solutions with bounded stored energy. The rates of growth and decay both depend only upon the geometry of the cylinder's cross-section, while the amplitude involves only data on the base of the cylinder. The special case of specified displacement on the base was considered in detail. Similar estimates for the compressible problem have been intensely studied, but almost all degenerate in the incompressible limit necessitating direct analyses of the present kind. The results derived here are immediately applicable to the corresponding problem in steady state Stokes flow along a pipe.

Practical applications to which the estimates might be relevant include bonding of rubber-like materials to the walls of a cylindrical hole drilled in a comparatively rigid metal, as occurs for example in certain sealant devices. The length of the hole need not be semi-infinite since the treatment presented is valid irrespective of the cylinder's length. Decay occurs in a finite cylinder provided the end opposite the base has either the displacement or traction specified pointwise zero, and the estimates might then be useful in certain contact problems. Other examples concern fibre reinforcement, geological intrusions, and more generally any situation requiring knowledge of edge effect penetration. It must be remarked, however, that similar estimates in compressible elasticity display a disparity when compared with known exact solutions and there is no reason to suppose the estimates derived here might possess greater accuracy.

The treatment may be extended to elastodynamics, to anisotropic incompressible elasticity, to other geometries, and to other constraints and other materials. The main purpose of this paper, however, has been to avoid such complexities and instead to emphasise essential features by application to a simple problem. Extensions are left for elsewhere.

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